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TECHNICAL NOTES

Thermal dispersion effects on non-Darcy convection over horizontal surfaces in saturated porous media

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INTRODUCTION

IN A RECENT paper [1], we have reported similarity solutions for steady, non-Darcy convection over a horizontal surface embedded in a saturated porous medium for the cases of natural, mixed and forced convection. As pointed out by Cheng [2] and Plumb [3], transverse thermal dispersion effects may become very important when inertial effects are prevalent. However, most previous studies [1, 4–7] on non-Darcy convection, either for natural or mixed convection, have not taken this effect into account. Only recently, Hong *et al.* [8] reported a study of non-Darcy natural convection along a vertical plate, in which thermal dispersion was included. Their results show that both inertia and dispersion are important at high Rayleigh numbers. Dispersion tends to enhance the heat transfer, while inertial effects decrease it. Whether heat transfer will increase, as compared to the Darcy case, depends on the balance between these mechanisms.

It is the purpose of this study to re-examine our previous results [1] by including thermal dispersion effects. Steady non-Darcy convection, in the form of natural, mixed and forced convection, is again considered for a heated horizontal surface embedded in a saturated porous medium. Under the assumed conditions, similarity solutions again exist for the case of constant surface heat flux.

ANALYSIS

Consider a two-dimensional non-Darcy flow over a horizontal, impermeable surface in a saturated porous medium. Having invoked the Boussinesq and boundary-layer approximations, the governing equations based on the Ergun formulation are given by

$$\frac{\partial^2\psi}{\partial y^2}+\frac{K'}{\nu}\frac{\partial}{\partial y}\left[\left(\frac{\partial\psi}{\partial y}\right)^2\right]=-\frac{Kg\beta}{\nu}\frac{\partial T}{\partial x} \tag{1}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}=\frac{\partial}{\partial y}\left[\alpha_e\frac{\partial T}{\partial y}\right] \tag{2}$$

where

$$\alpha_e=\alpha+\alpha'. \tag{3}$$

The thermal dispersion effect is introduced by assuming the effective thermal diffusivity α_e to have two components: α the molecular diffusivity and α' the diffusivity due to thermal dispersion. Following the linear model proposed by Plumb [3], the dispersion diffusivity is proportional to the stream-wise velocity component, that is

$$\alpha'=Cud \tag{4}$$

where C is a constant, which has a value ranging from 1/7 to 1/3. However, it has been pointed out by Hunt and Tien [9] that the solution obtained by assuming a constant dispersion coefficient and neglecting the wall effects will provide an upper bound for the estimation of heat transfer results.

The corresponding boundary conditions are

$$y=0,\quad T_w=T_\infty+Ax^\lambda,\quad v=0 \tag{5}$$

$$y\to\infty,\quad T=T_\infty,\quad u=0\,\text{(natural convection)} \tag{6a}$$

$$u=U_\infty=Bx^m\,\text{(mixed convection)}. \tag{6b}$$

With the appropriate transformation, equations (1) and (2) can be further reduced to a set of ordinary differential equations for which exact solutions are possible. The suitable similarity variables for such a transformation are those employed in the previous study [1], namely:

for natural convection

$$\eta=(Ra)^{1/3}\frac{y}{x} \tag{7}$$

$$\psi=\alpha(Ra)^{1/3}f(\eta); \tag{8}$$

for mixed convection

$$\eta=(Pe)^{1/2}\frac{y}{x} \tag{9}$$

$$\psi=\alpha(Pe)^{1/2}f(\eta). \tag{10}$$

Natural convection

After transformation, the resulting equations are given by

$$f''+\frac{K'\alpha}{\nu}\left[\frac{Kg\beta A}{\nu\alpha}\right]^{2/3}x^{(2\lambda-1)/3}[(f')^2]'+\lambda\theta+\frac{\lambda-2}{3}\eta\theta'=0 \tag{11}$$

$$\lambda f'\theta-\frac{\lambda+1}{3}f\theta'=\theta''+Cdx^{(2\lambda-1)/3}\left[\frac{Kg\beta A}{\nu\alpha}\right]^{2/3}(f''\theta'+f'\theta''). \tag{12}$$

These equations will be independent of x if $\lambda=1/2$, that is,

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NOMENCLATURE

- A constant defined in equation (5)
- B constant defined in equation (6b)
- C coefficient of the thermal dispersion effect
- d mean particle diameter or pore diameter
- Er Ergun number, $K'\alpha/d\nu$
- f dimensionless stream function
- g acceleration of gravity
- h local heat transfer coefficient
- k_e effective thermal conductivity of the saturated porous medium
- K permeability
- K' inertial coefficient of the Ergun equation
- Nu local Nusselt number, hx/k
- p pressure
- Pe Peclet number, $U_\infty x/\alpha$
- Pe_d Peclet number based on the pore diameter, $U_\infty d/\alpha$
- Ra Rayleigh number, $Kg\beta(T_w - T_\infty)x/\alpha\nu$
- Ra_d Rayleigh number based on the pore diameter, $Kg\beta A d^{3/2}/\alpha\nu$
- T temperature
- T_w surface temperature
- T_∞ ambient temperature or free stream temperature
- u, v velocity components in the x- and y-directions
- U free stream velocity
- x, y Cartesian coordinates.

Greek symbols

- α_e effective thermal diffusivity
- α molecular thermal diffusivity
- α' thermal diffusivity due to dispersion effect
- β thermal expansion coefficient of fluid
- η independent similarity variable
- θ dimensionless temperature, $(T - T_\infty)/(T_w - T_\infty)$
- λ constant defined in equation (5)
- μ dynamic viscosity
- ν kinematic viscosity
- ρ fluid density
- ψ stream function, equations (8) and (10).

Subscripts

- e effective properties
- mx mixed convection
- nc natural convection
- 0 properties related to Darcy flows.

the surface temperature varies with $x^{1/2}$. Thus, equations (11) and (12) can be simplified to

$$f'' + Er(Ra_d)^{2/3}[(f')^2]' + \frac{\theta}{2} - \frac{\eta}{2}\theta' = 0 \qquad (13)$$

$$\tfrac{1}{2}(f'\theta - f\theta') = \theta'' + C(Ra_d)^{2/3}(f'\theta'' + f''\theta') \qquad (14)$$

where Ra_d is the modified Rayleigh number based on the mean pore diameter. The grouping Er is a new dimensionless parameter which we tentatively name the Ergun number. It is defined as $Er = K'\alpha/d\nu$. The Ergun number characterizes the porous system under investigation since it represents the structure of the porous matrix, K'/d, and the thermophysical properties of the porous medium, α/ν. This parameter is a direct measurement of inertial effects, and its importance has also been recognized by Prasad and Tuntomo [10], in a study of inertial effects on natural convection in a vertical cavity.

The corresponding boundary conditions are

$$\eta = 0, \quad \theta = 1, \quad f = 0 \qquad (15)$$

$$\eta \to \infty, \quad \theta = 0, \quad f' = 0. \qquad (16)$$

Mixed convection

With the similarity variables given in equations (9) and (10), the governing equations are transformed to

$$f'' + \frac{K'B}{\nu}x^m[(f')^2]' = -\frac{Kg\beta A}{\nu B}\left[\frac{\alpha}{B}\right]^{1/2}\left[\frac{m-1}{2}\eta\theta' + \lambda\theta\right]x^{(2\lambda-1-3m)/2} \qquad (17)$$

$$\lambda\theta f' - \frac{m+1}{2}f\theta' = \theta'' + C\frac{U_\infty d}{\alpha}(f''\theta' + f'\theta''). \qquad (18)$$

They will be independent of x if $m = 1$ and $\lambda = 1/2$, that is, a uniform flow over a horizontal surface for which the surface temperature varies with $x^{1/2}$. Thus, the above equations can be simplified to

$$f'' + Er\,Pe_d[(f')^2]' = \frac{1}{2}\frac{Ra}{Pe^{3/2}}(\eta\theta' - \theta) \qquad (19)$$

$$\theta'' + C\,Pe_d(f'\theta'' + f''\theta') = \tfrac{1}{2}(f'\theta - f\theta') \qquad (20)$$

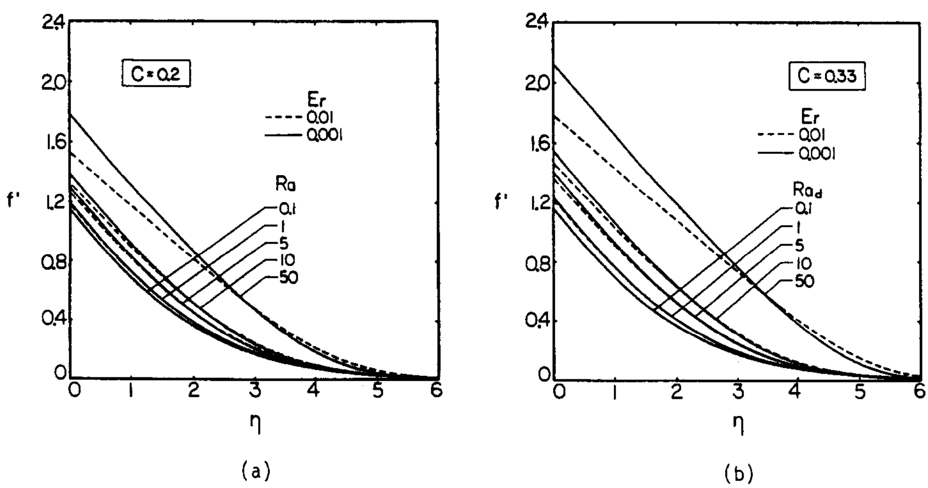

FIG. 1. Dimensionless velocity profiles for non-Darcy natural convection with thermal dispersion effects.

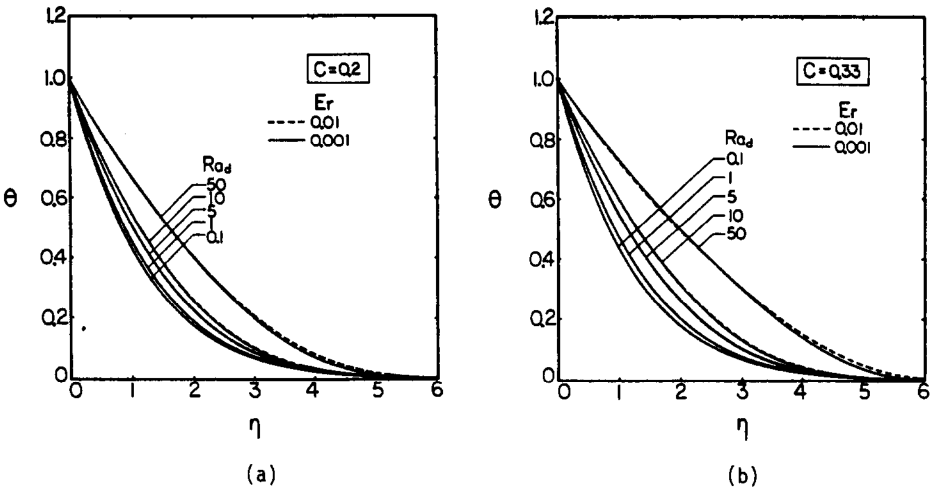

FIG. 2. Dimensionless temperature profiles for non-Darcy natural convection with thermal dispersion effects.

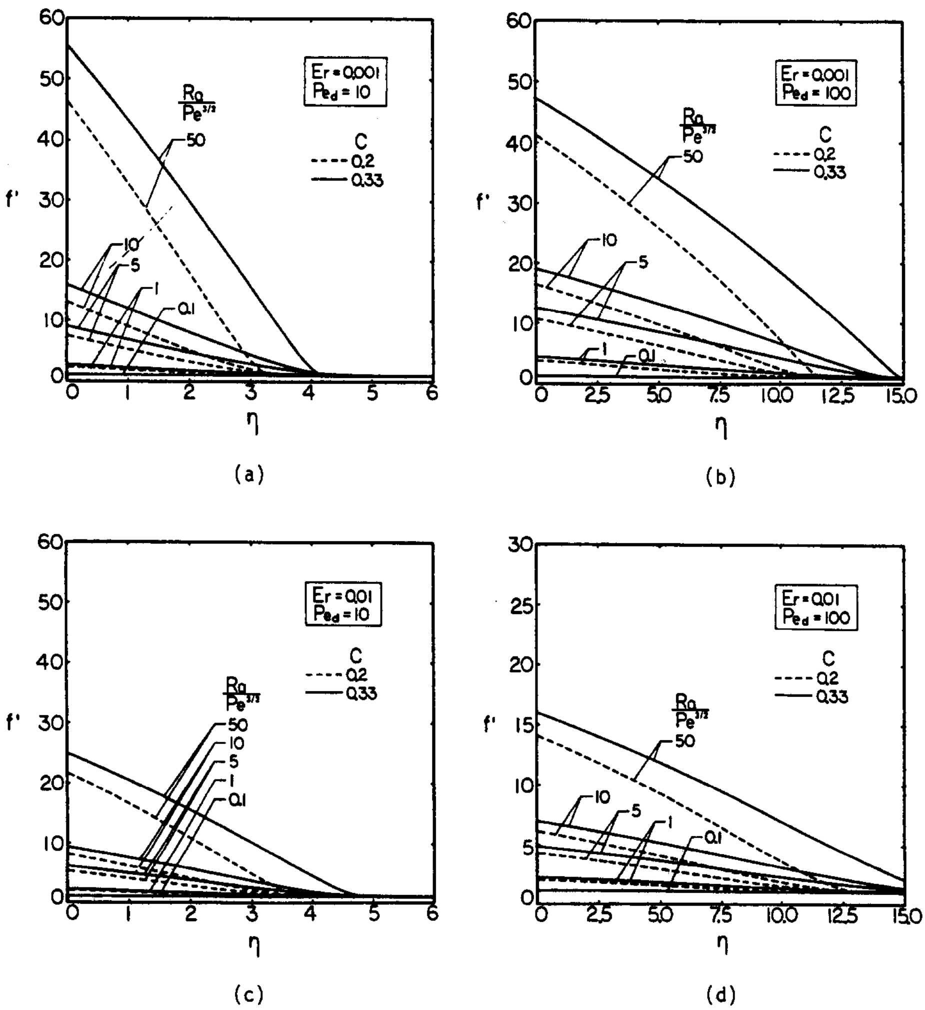

FIG. 3. Dimensionless velocity profiles for non-Darcy mixed convection with thermal dispersion effects.

where Pe_d is the Peclet number based on the mean pore diameter. The corresponding boundary conditions are

$$\eta = 0, \quad \theta = 1, \quad f = 0 \tag{21}$$

$$\eta \to \infty, \quad \theta = 0, \quad f' = 1. \tag{22}$$

Forced convection

It is noted that the governing equations for forced convection can be derived from equations (19) and (20) by simply setting $Ra/(Pe)^{3/2} = 0$. Therefore, they are given by

$$f'' + Er\,Pe_d[(f')^2]' = 0 \tag{23}$$

$$\theta'' + C\,Pe_d(f'\theta'' + f''\theta') = \tfrac{1}{2}(f'\theta - f\theta'). \tag{24}$$

RESULTS AND DISCUSSION

The transformed ordinary differential equations, with the corresponding boundary conditions, are solved by numerical integration using the Runge–Kutta method and the shooting technique with a systematic guessing of $\theta'(0)$ and $f'(0)$. The resulting profiles of dimensionless velocity and temperature are shown in Figs. 1 and 2 for natural convection, and in Figs. 3 and 4 for mixed convection.

Inertial and thermal dispersion effects on natural convection can be observed in Figs. 1 and 2. For a small Rayleigh number Ra_d, the velocity profiles, also the temperature profiles, are almost identical for all Ergun numbers, which means that the inertia effect is negligible at small Rayleigh numbers. As the Rayleigh number increases, a larger Ergun number will lead to a thicker hydrodynamic and thermal boundary layer. The same relation is found for the dispersion coefficient since thermal dispersion is very closely related to the inertia effect. This can be verified from the figures that the thermal dispersion has a more pronounced effect at a higher Rayleigh number, exactly where initial effects are prevalent.

For mixed convection, it is found that for a fixed Ergun number, a higher Peclet number Pe_d tends to thicken the hydrodynamic and thermal boundary layer. On the other hand, for a fixed Peclet number, a larger Ergun number also leads to a thicker boundary layer but a smaller slip velocity at the wall, i.e. $f'(0)$. Thermal dispersion effects display a similar trend. However, it is observed that there exists a basic difference between the velocity and temperature profiles. For velocity profiles, the boundary-layer thickness increases with the governing parameter $Ra/(Pe)^{3/2}$, while it decreases for temperature profiles.

It is interesting to note that the solutions of equations (23) and (24) turn out to be dependent on C and Pe_d only. The Ergun number has no influence over the heat transfer results. The reason is that for forced convection, the analysis is based on a fixed free stream velocity since the governing equations are derived from the mixed convection results. Therefore, inertial effects will show up in the total pressure gradient,

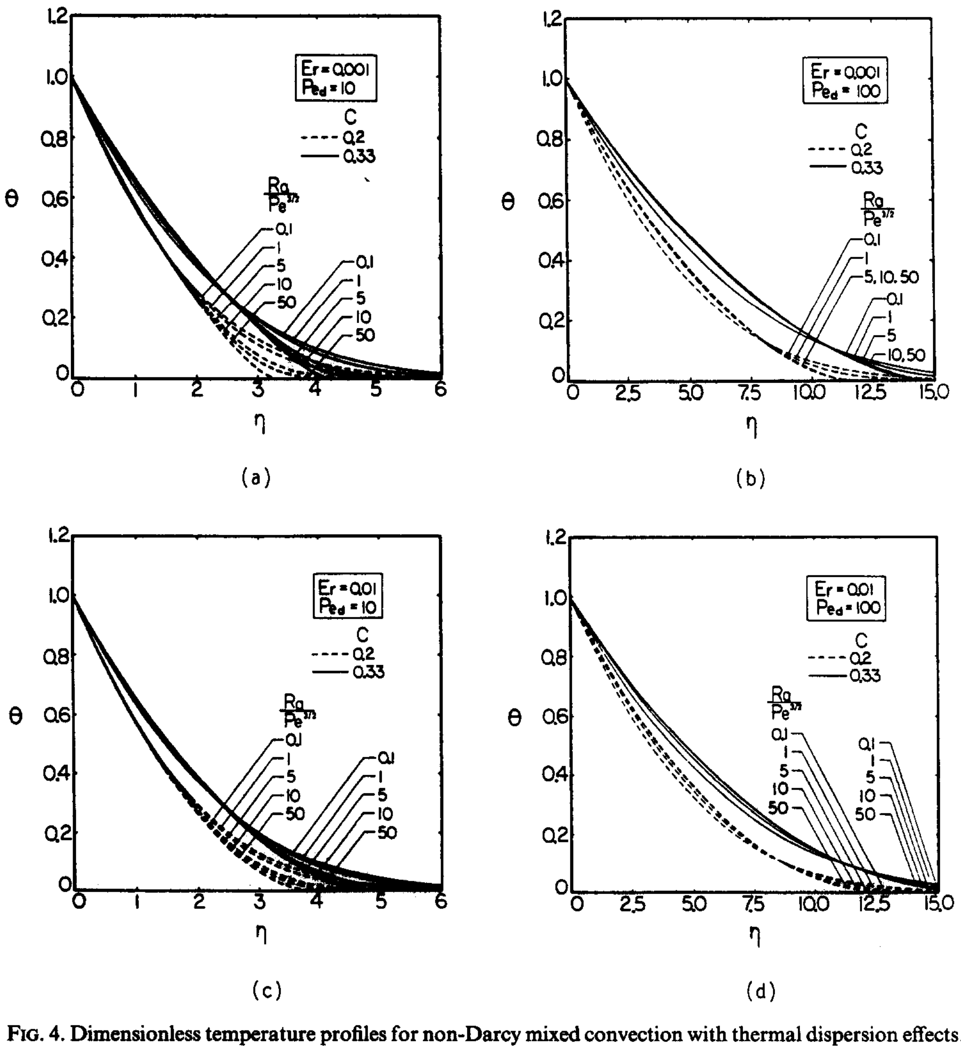

FIG. 4. Dimensionless temperature profiles for non-Darcy mixed convection with thermal dispersion effects.

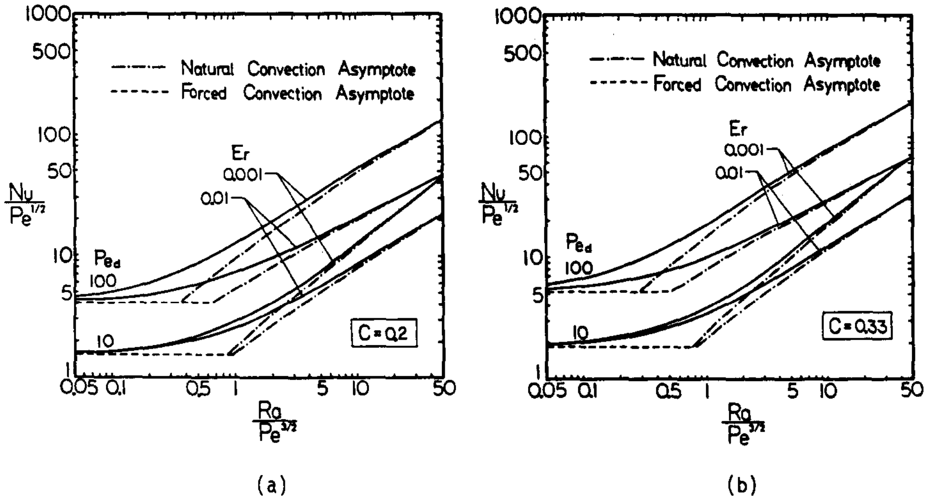

FIG. 5. Heat transfer results for non-Darcy mixed convection with thermal dispersion effects.

instead of the heat transfer results. This point is also stated in the previous study [1].

When the dispersion effect is taken into consideration, the local heat transfer rate is computed from

$$q = -k_e \frac{\partial T}{\partial y}\bigg|_{y=0} \tag{25}$$

where k_e is the effective thermal conductivity of the porous medium. With the definition of the Nusselt number, the heat transfer coefficient can be expressed as:

natural convection

$$\frac{Nu}{Ra^{1/3}} = -[1+C(Ra_d)^{2/3}f'(0)]\theta'(0) = [-\theta_e'(0)]_{nc}; \tag{26}$$

mixed convection

$$\frac{Nu}{Pe^{1/2}} = -[1+C\,Pe_d\,f'(0)]\theta'(0) = [-\theta_e'(0)]_{mx}; \tag{27}$$

forced convection

$$\frac{Nu}{Pe^{1/2}} = -[1+C\,Pe_d]\theta'(0). \tag{28}$$

Each expression comprises two terms, the first is the contribution due to molecular diffusion and the second is that due to dispersion. Though the value of $-\theta'(0)$, i.e. the contribution by the molecular diffusion, decreases with an increase of $Ra/(Pe)^{3/2}$ for mixed convection and Ra_d for natural convection, overall heat transfer is increased due to a significant contribution from dispersion.

Equation (27) is plotted in Fig. 5 as a function of $Ra/(Pe)^{3/2}$. The limiting cases of free and forced convection are also shown as asymptotes. For a fixed dispersion coefficient, the heat transfer coefficient increases with the Peclet number Pe_d, while it decreases with the Ergun number, which clearly shows that inertial effects tend to reduce the heat transfer. For a fixed Ergun number, the heat transfer coefficient increases with both Peclet number and the thermal dispersion coefficient, which indicates that dispersion will enhance the heat transfer. The corresponding asymptotes are obtained in the same manner as described before [1], which consists of two steps. First, rewrite equation (27) as

$$\frac{Nu}{Pe^{1/2}} = \frac{Nu}{Ra^{1/3}}\left[\frac{Ra}{Pe^{3/2}}\right]^{1/3}$$

$$= \left[\frac{Ra}{Pe^{3/2}}\right]\{-[1+C(Ra_d)^{2/3}f'(0)]\theta'(0)\}. \tag{29}$$

Second, apply the relation between $Er(Ra_d)$ and $Er\,Pe_d$

$$Er(Ra_d)^{2/3} = Er\,Pe_d\left(\frac{Ra_d}{Pe_d^{3/2}}\right)^{2/3} = Er\,Pe_d\left(\frac{Ra}{Pe^{3/2}}\right)^{2/3}. \tag{30}$$

With a given $Er\,Pe_d$ and $Ra/(Pe)^{3/2}$, $Er(Ra_d)^{2/3}$ is determined from equation (30). Once $Er(Ra_d)^{2/3}$ is specified, $[-\theta_e'(0)]_{nc}$ can be solved from equations (13) and (14). Therefore, the free convection asymptote is obtained, from equation (29), for each corresponding $Ra/(Pe)^{3/2}$.

The results can be best presented by the ratio of the heat transfer coefficient for the non-Darcy flow to that for the Darcy flow, i.e. $\theta_e'(0)/\theta_0'(0)$. Figures 6 and 7 present these ratios for free and mixed convection, respectively. As seen, the ratio is always greater than unity. This is quite different from our previous results [1] for which the ratio is always less than unity when only the inertial effect is considered. Thus, thermal dispersion can have an important role to play in determining the heat transfer coefficient. Its effect at high

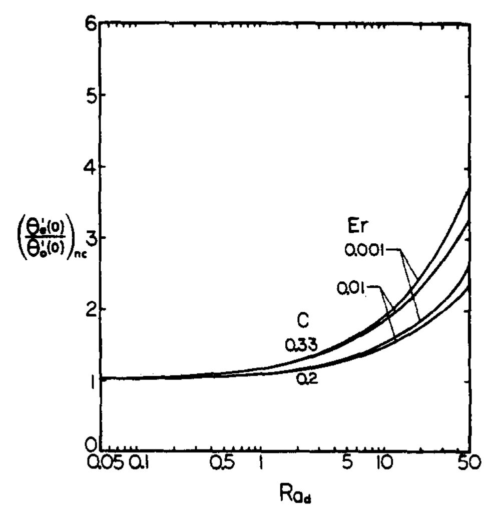

FIG. 6. The ratios of heat transfer coefficient of non-Darcy natural convection to that of Darcy flow.

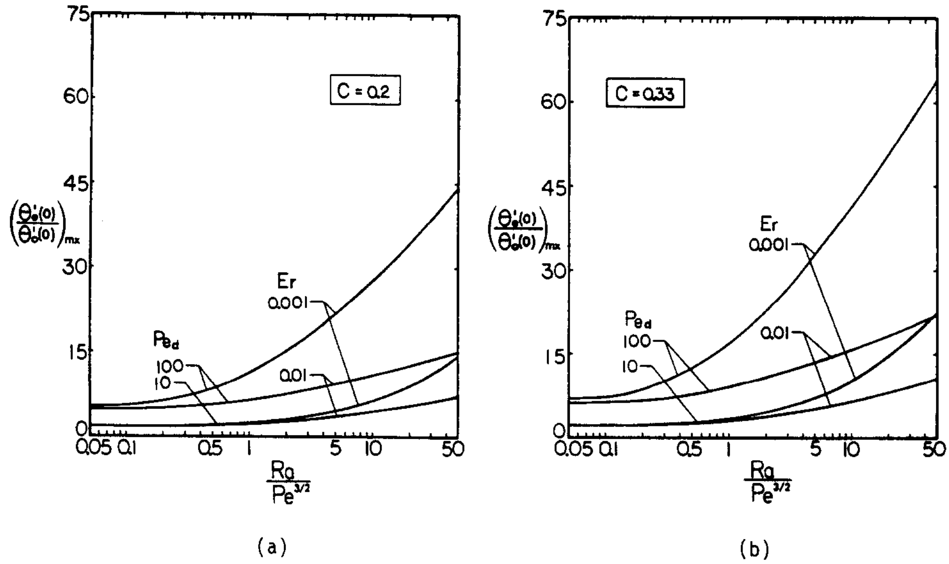

FIG. 7. The ratios of heat transfer coefficient of non-Darcy mixed convection to that of Darcy flow.

Rayleigh numbers needs to be considered carefully for an accurate estimate of heat transfer coefficients.

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